Solution to Problems ♠–7

Problem A: *Given real numbers a, b such that $1 \le a < b$. Show that*

$$\frac{e^a}{e^b} > \frac{a^a}{b^b},$$

where e is the base of natural logarithm.

Answer: The inequality

$$\frac{e^a}{e^b} > \frac{a^a}{b^b}$$

is equivalent to inequality

$$\left(\frac{e}{a}\right)^a > \left(\frac{e}{b}\right)^b.$$

Let $y = \left(\frac{e}{x}\right)^x$. Using the logarithmic differentiation we get

$$\frac{dy}{dx} = -\left(\frac{e}{x}\right)^x \cdot \ln x < 0, \qquad \text{if } x > 1.$$

Thus, for $x \ge 1$, $f(x) = \left(\frac{e}{x}\right)^x$ is a decreasing function, so $f(a) > f(b)$ and $\frac{e^a}{e^b} > \frac{a^a}{b^b}$.

No Correct solution received

Problem B: *Assume that $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions such that $f(g(x)) = g(f(x))$ for all x. Suppose that there is a real number x_0 such that $f(f(x_0)) = g(g(x_0))$. Show that then the equation $f(x) = g(x)$ has a solution.*

Answer: Suppose towards contradiction that the equation $f(x) = g(x)$ has no solutions.

Consider the function

$$h(x) = f(x) - g(x), \qquad x \in \mathbb{R}.$$

Then h is continuos and has no zeroes and consequently either it assumes only positive values, or it assumes only negative values. Therefore for any $y, z \in \mathbb{R}$ we have $h(y) + h(z) \neq 0$. Letting $y = f(x_0)$ and $z = g(x_0)$ we conclude that

$$\begin{aligned} 0 &\neq h\big(f(x_0)\big) + h\big(g(x_0)\big) = \\ &= f\big(f(x_0)\big) - g\big(f(x_0)\big) + f\big(g(x_0)\big) - g\big(g(x_0)\big) = \\ &= f\big(f(x_0)\big) - g\big(g(x_0)\big). \end{aligned}$$

A contradiction.

Correct solution were received from :

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